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The capillary boundary layer for standing waves

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The linear, free-surface oscillations of an inviscid fluid in a cylindrical basin subject to the contact-line condition $c\boldsymbol{n}\cdot\boldsymbol{\nabla}\zeta = \zeta_t$ (ζ is the free-surface displacement and c is a complex constant) are determined through a boundary-layer approximation for $l/a \ll 1$, where a is a characteristic length of the cross-section and l is the capillary length. The primary result is $\omega^2 = \omega_n^2[1+(l/a)\mathscr{F}(\zeta_n; c/\omega_n l)]$, where ω is the frequency of a free oscillation, ω_n is the natural frequency for a particular normal mode, $\zeta = \zeta_n$, in the limit $l/a \to 0$, and $\mathscr{F}(\zeta_n; c/\omega_n l)$ is a corresponding form factor. The imaginary part of $\mathscr{F}$ is positive (for the complex time dependence $\exp(\mathrm{i}\omega t)$) if $\mathrm{Re}(c) > 0$, which implies positive dissipation through contact-line motion. Explicit results are derived for circular and rectangular cylinders and compared with Graham-Eagle's (1983) results for the circular cylinder for $c = 0$ and Hocking's (1987) results for the two-dimensional problem. The exact eigenvalue equation for the circular cylinder and a variational approximation for an arbitrary cross-section are derived on the assumption that the static meniscus is negligible.

1. Introduction

The linear eigenvalue problem for inviscid gravity waves in a cylinder of cross-section S, lateral boundary L, and uniform depth h is described by

$$\nabla^2\phi = 0 \quad (\boldsymbol{x} \text{ in } S, \quad -h < z < 0), \tag{1.1}$$

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\phi = 0 \quad \text{on} \quad L, \quad \phi_z = 0 \quad \text{on} \quad z=-h, \tag{1.2a, b}$$

$$\phi_z = \mathrm{i}\omega\zeta, \quad \mathrm{i}\omega\phi + g\zeta = T\nabla^2\zeta \quad (z=0), \tag{1.3a, b}$$

where ϕ and ζ are the complex amplitudes of the velocity potential and free-surface displacement, both of which contain the implicit factor $\exp(\mathrm{i}\omega t)$, $\boldsymbol{n}$ is the inwardly directed normal to L, $T \equiv$ surface tension/density, and the frequency ω is to be determined.

If $T = 0$ the problem is well posed without the prior specification of a boundary condition for ζ on L, the substitution

$$\phi(\boldsymbol{x}, z) = \mathrm{i}\omega\zeta(\boldsymbol{x})\frac{\cosh k(z+h)}{k\sinh kh} \tag{1.4}$$

reduces (1.1)–(1.3a) to the classical eigenvalue problem (Lamb 1932, §257).

$$\nabla^2\zeta + k^2\zeta = 0 \quad \text{in} \quad S, \quad \boldsymbol{n}\cdot\boldsymbol{\nabla}\zeta = 0 \quad \text{on} \quad L, \tag{1.5a, b}$$

and (1.3b) then yields the gravity-wave dispersion relation $\omega^2 = gk\tanh kh$. But if $T > 0$ the boundary condition on ζ generally differs from (1.5b); following Hocking (1987), I assume

$$c\boldsymbol{n}\cdot\boldsymbol{\nabla}\zeta = \mathrm{i}\omega\zeta \quad \text{on} \quad L, \tag{1.6}$$

where c is a velocity that Hocking assumes to be real but I assume to be complex (since $\nabla\zeta$ need not be in phase with $\mathrm{i}\omega\zeta$). The essential parameters of the problem then are

$$\kappa \equiv kl, \quad \gamma \equiv \frac{c}{\omega l}, \tag{1.7a, b}$$

where k is an eigenvalue of (1.1)–(1.3) and (1.6), and

$$l \equiv (T/g)^{\frac{1}{2}} \tag{1.8}$$

is the capillary length. The requirement that the damping associated with contact-line motion be positive implies $\mathrm{Re}(\gamma) > 0$.

Experimental determinations of γ do not appear to be available, while theoretical considerations (Miles 1990) suggest that $|\gamma| \ll 1$, for harmonic motion. Benjamin & Scott (1979) have proposed and experimentally confirmed that $\gamma = 0$ for waves in a brim-full container; however, observation suggests that contact-line motion contributes significantly to the damping of standing waves at a protuding wall. This motion may be nonlinear, but that can be determined only by experiment. The present analysis, in which (1.6) may be regarded as the most general phenomenological hypothesis that is compatible with linearity and the dynamical boundary condition (1.3b), is directed towards such experiments and towards a better understanding of the damping of waves in closed containers, which typically exceeds that inferred from viscous boundary-layer calculations (Miles 1967).

The problem posed by (1.1)–(1.3) and (1.6) for $\gamma = 0$ ($\zeta = 0$ on L) is considered by Benjamin & Scott (1979), who are concerned primarily with waves propagating along a rectangular channel but give a variational formulation for standing waves, and by Graham-Eagle (1983), who obtains explicit results for a circular cylinder. The two-dimensional problem is solved by Hocking (1987) for $\gamma > 0$, but his analytical results are valid for complex γ. All of these solutions neglect the static meniscus, although Benjamin & Scott consider the meniscus in an appendix.

I attack the eigenvalue problem posed by (1.1)–(1.3) and (1.6) on the assumption that $\kappa \ll 1$. This restriction justifies the neglect of the static meniscus, the inclusion of which would require that (Miles 1990): (i) the free-surface boundary conditions (1.3) be imposed at the meniscus, $z = z_{\mathrm{m}}(\boldsymbol{x})$, rather than $z = 0$; (ii) $\nabla z_{\mathrm{m}} \cdot \nabla\phi$ be added to the right-hand side of (1.3a); (iii) $\nabla^2\zeta$ be replaced by $\nabla \cdot (p\nabla\zeta)$ in (1.3b), where $p = p(\nabla z_{\mathrm{m}})$. The error factor associated with this neglect is $1 + O(\kappa)$.

The restriction $\kappa \ll 1$ also suggests that the effects of the contact-line condition (1.6), vis-à-vis (1.5b), are confined to a capillary boundary layer of characteristic thickness l. This hypothesis is supported by the remark that (1.3b), qua differential equation for ζ in which the hydrodynamic pressure $\mathrm{i}\omega\phi$ (after factoring out the density) acts as the forcing function, admits a complementary solution that decays away from the wall with an e-folding length l. I carry out the boundary-layer analysis in §2 to obtain

$$\frac{\omega^2 - \omega_n^2}{\omega_n^2} = \frac{\mathscr{F}_n}{1 - \mathrm{i}\gamma}[1 + O(\kappa)], \tag{1.9}$$

where $\mathscr{F}_n = O(\kappa)$ is a form factor (see (2.12)), and ω_n is the natural frequency for a particular eigensolution of (1.5) in the limit $\kappa \downarrow 0$. The viscous correction to $\omega^2 - \omega_n^2$ may be superimposed on (1.9) provided that both $\kappa \ll 1$ and $kl_\nu \ll 1$, where $l_\nu = (2\nu/\omega)^{\frac{1}{2}}$ is the thickness of the viscous boundary layer.

The error factor $1 + O(\kappa)$ implied by the neglect of the meniscus in the present formulation suggests that there may be little profit in improving the boundary-layer approximation (1.9). However, it appears from an analysis of the corresponding problem for an oscillating plate (Miles 1990) that the meniscus correction may be numerically small, and I therefore carry out an integral-equation formulation of the boundary-value problem in §3 without the restriction $\kappa \ll 1$. This formulation yields the exact eigenvalue equation for a circular cylinder (or for any other cross-section for which L is a level surface in coordinates for which (1.5) is separable) and leads to a generalization of the variational formulation of Benjamin & Scott (1979) for $\gamma = 0$ to arbitrary γ, which I develop in Appendix A; however, its primary value in the present context is the provision of error estimates for the boundary-layer approximation.

I consider, as examples, circular and rectangular cylinders in §§4 and 5. The approximation (1.9) is within 6% of Graham-Eagle's (1983) numerical results for $\gamma = 0$ and $l/a < 0.4$ for the dominant axisymmetric mode in a circular cylinder of radius a; it is within 10% of the corresponding result for the dominant antisymmetric mode for $l/a < 0.2$. These comparisons provide confidence in the boundary-layer approximation for $\kappa \ll 1$, but only experiment can provide a check on the predicted effects of γ, including, in particular, contact-line damping.

2. Boundary-layer approximation

We pose the solution of (1.1) and (1.2*a*, *b*) in the form

$$\phi = \sum_n A_n \zeta_n(\boldsymbol{x}) \frac{\cosh k_n(z+h)}{\cosh k_n h}, \tag{2.1}$$

where $\{\zeta_n(\boldsymbol{x}); k_n\}$ is a complete, orthogonal set of eigenfunctions of (1.5). The index n is, in general, an abbreviation for a pair of indices, and the summation in (2.1) then is over a doubly infinite, discrete spectrum – e.g. $\zeta_{mn} = \cos(m\pi x/a)\cos(n\pi y/b)$ for a rectangle.

Substituting (2.1) into (1.3*b*), we obtain

$$\mathscr{L}\zeta \equiv T\nabla^2\zeta - g\zeta = \mathrm{i}\omega \sum_n A_n \zeta_n(\boldsymbol{x}), \tag{2.2}$$

a particular solution of which is given by (recall that $\nabla^2\zeta_n = -k_n^2\zeta_n$)

$$\zeta_p = \sum_n B_n \zeta_n(\boldsymbol{x}), \quad B_n \equiv \frac{-\mathrm{i}\omega A_n}{g+Tk_n^2}. \tag{2.3a, b}$$

The contact-line condition (1.6) requires ζ_p to be complemented by a solution of $\mathscr{L}\zeta = 0$ that, by hypothesis, is significant only in a boundary layer of characteristic thickness l on L. Let $(\mathscr{x}, \mathscr{y})$ be boundary-layer coordinates that are (normal, tangential) to L and for which the characteristic lengths are $(l, l/\kappa)$, where $l \equiv (T/g)^{\frac{1}{2}}$ is the capillary length; then

$$\nabla^2\zeta = \zeta_{\mathscr{x}\mathscr{x}}[1+O(\kappa^2)], \tag{2.4}$$

and the complementary solution of (2.2) has the form

$$\zeta_c = \mathrm{e}^{-\mathscr{x}/l} f(\mathscr{y}), \tag{2.5}$$

where f is an arbitrary function that is slowly varying relative to $\exp(-\mathscr{x}/l)$, and, here and subsequently, error factors of $1+O(\kappa^2)$ are implicit. We determine f by requiring $\zeta_p + \zeta_c$ to satisfy (1.6) and invoking $\zeta_n \equiv \zeta_n(\mathscr{y})$ and $\boldsymbol{n}\cdot\nabla\zeta_n = 0$ on L. The end result for ζ, after introducing $\gamma \equiv c/\omega l$, is

$$\zeta = \zeta_p + \zeta_c = \sum_n B_n[\zeta_n(\boldsymbol{x}) - (1-\mathrm{i}\gamma)^{-1}\mathrm{e}^{-\mathscr{x}/l}\zeta_n(\mathscr{y})]. \tag{2.6}$$

Combining (2.1) and (2.6) in (1.3*a*) and eliminating A_n through (2.3*b*), we obtain

$$\sum_n B_n[(\omega^2-\omega_n^2)\,\zeta_n(\boldsymbol{x})-(1-\mathrm{i}\gamma)^{-1}\,\omega^2\,\mathrm{e}^{-x/l}\zeta_n(y)] = 0, \tag{2.7}$$

where

$$\omega_n^2 \equiv (gk_n+Tk_n^3)\tanh k_n h. \tag{2.8}$$

Multiplying (2.7) through by $\zeta_m(\boldsymbol{x})$, integrating over S, and invoking the orthogonality of the ζ_n and the approximation [which follows from $\zeta_m(\boldsymbol{x}) = \zeta_m(y)+O(\kappa^2)$ for $x = O(l)$]

$$\iint \zeta_m(\boldsymbol{x})\,\zeta_n(y)\,\mathrm{e}^{-x/l}\,\mathrm{d}S = l\int \zeta_m\,\zeta_n\,\mathrm{d}y = O\left(\kappa\iint \zeta_n^2\,\mathrm{d}S\right), \tag{2.9}$$

we obtain the infinite set of linear equations

$$[C_{mn}][B_n] = 0, \tag{2.10a}$$

where

$$C_{mn} = \delta_{mn}(\omega^2-\omega_n^2)\iint \zeta_n^2\,\mathrm{d}S-(1-\mathrm{i}\gamma)^{-1}\,\omega^2 l\int \zeta_m\,\zeta_n\,\mathrm{d}L, \tag{2.10b}$$

δ_{mn} is the Kronecker delta, and $\mathrm{d}L \equiv \mathrm{d}y$.

It follows from (2.9) and (2.10*b*) that the roots of the determinantal equation $|C_{mn}| = 0$ are given by†

$$\frac{\omega^2-\omega_n^2}{\omega_n^2} = \frac{\mathscr{F}_n}{1-\mathrm{i}\gamma}+O(\kappa^2), \tag{2.11}$$

where

$$\mathscr{F}_n \equiv \frac{l\displaystyle\int \zeta_n^2\,\mathrm{d}L}{\displaystyle\iint \zeta_n^2\,\mathrm{d}S} = O(\kappa) \tag{2.12}$$

is a form factor for $\zeta_n(\boldsymbol{x})$.

3. Integral-equation formulation

We define the finite-Fourier transform and its inverse corresponding to $\{\zeta_n(\boldsymbol{x});k_n\}$ by

$$F_n = \iint f(\boldsymbol{x})\,\zeta_n(\boldsymbol{x})\,\mathrm{d}S, \quad f(\boldsymbol{x}) = \sum_n \frac{F_n\,\zeta_n(\boldsymbol{x})}{N_n}, \quad N_n = \iint \zeta_n^2\,(\boldsymbol{x})\,\mathrm{d}S. \tag{3.1a–c}$$

Fourier-transforming (1.1), (1.2*b*) and (1.3*a*, *b*) with the aid of Green's theorem,

$$\iint \zeta_n\,\nabla^2 f\,\mathrm{d}S = \iint f\,\nabla^2\zeta_n\,\mathrm{d}S-\int (\boldsymbol{n}\cdot\boldsymbol{\nabla} f)\,\zeta_n\,\mathrm{d}L, \tag{3.2}$$

where $\boldsymbol{n}$ is the inwardly directed normal to L, and invoking (1.2*a*) and (1.6), we obtain

$$\nabla^2\Phi_n-k_n^2\,\Phi_n = 0 \quad (-h<z<0), \tag{3.3}$$

$$\Phi_{nz} = 0 \quad (z=-h), \tag{3.4}$$

$$\Phi_{nz} = \mathrm{i}\omega Z_n, \quad \mathrm{i}\omega\Phi_n+(g+Tk_n^2)\,Z_n = -\frac{\mathrm{i}\omega T}{c}\int \zeta\zeta_n\,\mathrm{d}L \quad (z=0), \tag{3.5a, b}$$

where Φ_n and Z_n are the transforms of ϕ and ζ. The solution of (3.3)–(3.5) is given by

$$\Phi_n = \mathrm{i}\omega Z_n\frac{\cosh k_n(z+h)}{k_n\sinh kh}, \quad Z_n = \left(\frac{\mathrm{i}\omega T}{c}\right)\left[\frac{k_n\tanh kh}{\omega^2-\omega_n^2}\right]\int \zeta\zeta_n\,\mathrm{d}L, \tag{3.6a, b}$$

† A consideration of the next approximation implies that $O(\kappa^2)$ comprises $O(\kappa^2\ln\kappa)$ in (2.11). See e.g. §5.2.

where ω_n^2 is given by (2.8). Inverting (3.6*b*) through (3.1*b*), invoking $\omega T/c = gl/\gamma$ and $\zeta_n \equiv \zeta_n(\mathscr{y})$ on L, and choosing $\boldsymbol{x}$ on L, we obtain the integral equation

$$\int_L G(\mathscr{y}, \eta)\,\zeta(\eta)\,\mathrm{d}\eta = \mathrm{i}\gamma\zeta(\mathscr{y}) \tag{3.7}$$

for the determination of ζ on L, where

$$G(\boldsymbol{x}, \mathscr{y}) = l\sum_n \frac{\zeta_n(\boldsymbol{x})\,\zeta_n(\mathscr{y})}{N_n\,\Omega_n}, \quad \Omega_n \equiv \frac{\omega_n^2-\omega^2}{gk_n\tanh k_n h}. \tag{3.8a, b}$$

We remark that (3.7) reduces to an exact eigenvalue equation if L is a level surface in a coordinate system for which (1.5) is separable; see e.g. §4.2.

4. Circular cylinder

4.1. *Boundary-layer approximation*

The normal modes for a circular cylinder of radius a are given by

$$\zeta_{mn} = J_m(k_{mn}\,r)\cos m\theta, \quad J'_m(k_{mn}\,a) = 0, \tag{4.1a, b}$$

where J_m is a Bessel function, r and θ are polar coordinates, m and n are integers, and $\cos m\theta$ may be replaced by any linear combination of $\cos m\theta$ and $\sin m\theta$. Substituting (4.1*a*) into (2.12), we obtain

$$\mathscr{F}_{mn} = \frac{2k_{mn}^2\,al}{k_{mn}^2\,a^2-m^2}. \tag{4.2}$$

We consider as examples the axisymmetric and dominant antisymmetric ($k_{11}\,a = 1.841$) modes, for which (2.11) and (4.2) yield

$$\frac{\omega^2-\omega_{0n}^2}{\omega_{0n}^2} = \frac{2\lambda}{1-\mathrm{i}\gamma}, \quad \frac{\omega^2-\omega_{11}^2}{\omega_{11}^2} = \frac{2.84\lambda}{1-\mathrm{i}\gamma}, \quad \lambda \equiv \frac{l}{a}. \tag{4.3a–c}$$

These approximations are compared with Graham-Eagle's (1983) results for $\gamma = 0$ in table 1.

4.2. *Exact eigenvalue equation*

Combining (3.8) and (4.1*a*) in the integral equation (3.7) and positing

$$\zeta = \zeta_m(a)\cos m\theta, \tag{4.4}$$

we find that $\zeta_m(a)$ may be cancelled to obtain the exact eigenvalue equation

$$g\sum_n \frac{\mathscr{F}_{mn}\,k_{mn}\tanh k_{mn}\,h}{\omega^2-\omega_{mn}^2} + \mathrm{i}\gamma = 0. \tag{4.5}$$

We recover the boundary-layer approximation by separating the term with the smallest denominator in (4.5), summing the remaining terms through the Euler-summation formula, and letting $l/a \downarrow 0$. We recover Graham-Eagle's (1983) results by letting $\gamma = 0$, $\tanh k_{mn}\,h = 1$, and $m = 0$ or 1 in (4.5). It may be inferred from (4.5) that (4.3*a*/*b*) over/under-estimates ω in this limiting ($\gamma \rightarrow 0$, $kh \rightarrow \infty$) case.

l/a	ω^2/ω^2_{0n} (4.3a)	ω^2/ω^2_{01}	ω^2/ω^2_{02}	ω^2/ω^2_{03}	ω^2/ω^2_{11}	ω^2/ω^2_{11} (4.3b)
0.100	1.200	1.200	1.201	1.190	1.303	1.284
0.224	1.45	1.48	1.37	1.29	1.83	1.64
0.316	1.63	1.63	1.43	1.31	2.30	1.90
0.447	1.89	1.77	1.46	1.32	2.95	2.27

TABLE 1. Frequency ratio ω^2/ω^2_{mn} for circular cylinder, as calculated by Graham-Eagle (1983) and from the boundary-layer approximation (4.3) for $\gamma = 0$.

5. Rectangular cylinder

5.1. *Boundary-layer approximation*

The normal modes for a rectangular cylinder with walls at $x = 0, a$ and $y = 0, b$ are given by

$$\zeta_{mn} = \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \quad k^2_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, \tag{5.1a, b}$$

where m and n are integers and ζ_{00} is excluded by conservation of mass. Substituting (5.1a) into (2.12), we obtain

$$\mathscr{F}_{mn} = \left[\frac{2(2-\delta_{0m})}{a} + \frac{2(2-\delta_{0n})}{b}\right] l. \tag{5.2}$$

The corresponding variational approximation is derived in Appendix A.

The two-dimensional result follows from the limit $b \uparrow \infty$ (which is not equivalent to $n = 0$ in the present context). Substituting $\mathscr{F}_m = 4l/a$ from (5.2) into (2.11), we obtain (cf. (4.3))

$$\frac{\omega^2 - \omega^2_m}{\omega^2_m} = \frac{2\lambda}{1 - \mathrm{i}\gamma}, \quad \lambda \equiv \frac{2l}{a}. \tag{5.3a, b}$$

5.2. *Exact two-dimensional eigenvalue equation*

The exact eigenvalue problem for the two-dimensional problem, for which

$$\zeta_m = \cos\left(\frac{m\pi x}{a}\right), \quad k_m = \frac{m\pi}{a} \quad (m = 1, 2, \ldots), \tag{5.4a, b}$$

may be derived as in §4.2 and is given by (cf. (4.5))

$$\frac{4gl}{a}\sum_m \frac{k_m \tanh k_m h}{\omega^2 - \omega^2_m} + \mathrm{i}\gamma = 0, \tag{5.5}$$

where m is summed over either the even or the odd integers.

Letting $\tanh k_m h = 1$ in (5.5) and introducing $\mathscr{m}$ through the identity

$$\frac{\omega^2}{gk_1} \equiv \mathscr{m}(1 + \kappa_1^2 \mathscr{m}^2), \quad \kappa_1 \equiv k_1 l, \tag{5.6a, b}$$

where (by definition) $\mathscr{m} \to m$ as $\kappa_1 \to 0$, we reduce (5.5) to Hocking's (1987) result for the deep-water problem in the form (after allowing for differences in notation)

$$M(\mathscr{m}) + \tfrac{1}{4}\mathrm{i}\pi\gamma = 0, \tag{5.7}$$

where

$$M(m) = \kappa_1 \sum_n \frac{n}{m(1+\kappa_1^2 m^2)-n(1+\kappa_1^2 n^2)} = \kappa_1 \sum_n \frac{n}{(m-n)[1+\kappa_1^2(m^2+mn+n^2)]}. \quad (5.8)$$

This series is summed in Appendix B. Invoking (B 8) and letting $m = 1$, we obtain the second approximation

$$\frac{\omega^2-\omega_1^2}{\omega_1^2} = \frac{2\lambda}{1-\mathrm{i}\gamma-\lambda[\ln(\pi\lambda)+\frac{5}{2}-\gamma]+O(\lambda^2)}, \quad (5.9)$$

where $\gamma = 0.5772\ldots$ is Euler's constant. The square-bracketed quantity is 0.765 for $\lambda = 0.1$.

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Appendix A. Variational approximation

Multiplying the integral equation (3.7) through by $\zeta(x)$ and integrating around L, we obtain the quadratic form

$$Q\{\zeta(x);\omega\} = \iint \zeta(y)\,G(y,\eta)\,\zeta(\eta)\,\mathrm{d}y\,\mathrm{d}\eta - \mathrm{i}\gamma\int \zeta^2(y)\,\mathrm{d}y = 0, \quad (\mathrm{A}\ 1)$$

which is stationary with respect to first-order variations of $\zeta(y)$ about the true solution(s) of (3.7).

Substituting the trial function $\zeta = \zeta_n$ into (A 1) and separating the corresponding term from the Green's function (3.8) on the hypothesis that $\omega^2-\omega_n^2 = O(\kappa\omega_n^2)$, we obtain

$$\frac{\omega^2-\omega_n^2}{\omega_n^2} = \frac{\mathscr{F}_n}{(1+k_n^2 l^2)(S_n-\mathrm{i}\gamma)}, \quad (\mathrm{A}\ 2)$$

where ω_n and $\mathscr{F}_n$ are defined by (2.8) and (2.12),

$$S_n = l\sum_m{}' \frac{\left(\int \zeta_n\zeta_m\,\mathrm{d}L\right)^2}{\Omega_m\left(\iint \zeta_m^2\,\mathrm{d}S\right)\left(\int \zeta_n^2\,dL\right)}, \quad (\mathrm{A}\ 3)$$

and the prime implies the omission of $m = n$ from the summation. We emphasize that S_n depends on ω through Ω_n, in consequence of which (A 2) does not determine ω explicitly; however, it appears that an iterative solution, starting from $\omega = \omega_n$, should converge quite rapidly.

Considering, for example, the dominant mode in a rectangular tank, for which $a > b$, $m = 1$, $n = 0$, and ζ_{mn} and $\mathscr{F}_{10}$ are given by (5.1) and (5.2), we obtain

$$\mathscr{F}_{10} = \frac{2(a+2b)\,l}{ab}, \quad S_{10} = \left(\frac{4l}{a+2b}\right)\left[\frac{a}{b}\sum_{n=1}^{\infty}\frac{1}{\Omega_{1,2n}} + \frac{2b}{a}\sum_{m=1}^{\infty}\frac{1}{\Omega_{2m+1,0}}\right], \quad (\mathrm{A}\ 4)$$

where Ω_{mn} and k_{mn} are given by (3.8*b*) and (5.1*b*).

Appendix B. Summation of $M(\mathscr{m})$

A partial-fraction expansion of the series (5.8) yields

$$M = \sum_n \left\{ \left(\frac{\kappa}{1+3\kappa^2} \right) \left(\frac{1}{\mathscr{m}-n} \right) - \frac{\kappa_1}{\rho} \mathrm{Im} \left[\left(\frac{\frac{1}{2}\kappa - \mathrm{i}\rho}{\frac{3}{2}\kappa - \mathrm{i}\rho} \right) \left(\frac{1}{(n+\frac{1}{2}\mathscr{m})\,\kappa_1 - \mathrm{i}\rho} \right) \right] \right\}, \tag{B 1}$$

where
$$\kappa \equiv \mathscr{m}\kappa_1, \quad \rho \equiv (1+\tfrac{3}{4}\kappa^2)^{\frac{1}{2}}, \tag{B 2a, b}$$

and n is summed over either the odd or the even integers. Invoking the recurrence formula (Abramowitz & Stegun 1964, §5.3),

$$\psi(n+1+z) = \psi(z) + \frac{1}{z} + \frac{1}{z+1} + \ldots + \frac{1}{z+n}, \tag{B 3}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, and the reflection formula (Abramowitz & Stegun 1964, §5.3)

$$\psi(1-z) = \psi(z) + \pi \cot \pi z, \tag{B 4}$$

we obtain

$$\sum_n^N \frac{1}{\mathscr{m}-n} = -\tfrac{1}{2}\psi(\tfrac{1}{2}N+1-\tfrac{1}{2}\mathscr{m}) + \tfrac{1}{2}\psi(1+\tfrac{1}{2}\mathscr{m}-\tfrac{1}{2}\nu) + \tfrac{1}{2}\pi \cot[\tfrac{1}{2}\pi(\mathscr{m}-\nu)] \tag{B 5}$$

and
$$\sum_n^N \frac{1}{n+\frac{1}{2}\mathscr{m}-\mathrm{i}\hat{\rho}} = \tfrac{1}{2}\psi(\tfrac{1}{2}N+1+\tfrac{1}{4}\mathscr{m}-\tfrac{1}{2}\mathrm{i}\hat{\rho}) - \tfrac{1}{2}\psi(\tfrac{1}{4}\mathscr{m}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mathrm{i}\hat{\rho}), \tag{B 6}$$

wherein n is summed over the odd (even) integers, N is odd (even), $\nu = 1(0)$, and $\hat{\rho} \equiv \rho/\kappa_1$. Substituting (B 5) and (B 6) into (B 1) and letting $N \uparrow \infty$ (in which limit the contributions of the N-dependent ψ-functions cancel), we obtain

$$M = \left(\frac{\kappa}{1+3\kappa^2} \right) \{ \tfrac{1}{2}\psi(1+\tfrac{1}{2}\mathscr{m}-\tfrac{1}{2}\nu) + \tfrac{1}{2}\pi \cot[\tfrac{1}{2}\pi(\mathscr{m}-\nu)] \}$$
$$+ \frac{1}{2\rho} \mathrm{Im} \left\{ \left(\frac{1+\frac{3}{2}\kappa^2 - \mathrm{i}\kappa\rho}{1+3\kappa^2} \right) \psi \left(\tfrac{1}{4}\mathscr{m} - \frac{\mathrm{i}\rho}{2\kappa_1} + \tfrac{1}{2}\nu \right) \right\}. \tag{B 7}$$

We now let $\kappa \to 0$, so that $\mathscr{m}$ tends to an integer m that has the same parity as the summation index n in (B 1), to obtain

$$M = \frac{m\kappa_1}{\mathscr{m}-m} - \tfrac{1}{4}\pi + \tfrac{1}{2}\kappa \left[\ln \kappa + \tfrac{1}{2} + \frac{1+\nu}{m} + \psi(1+\tfrac{1}{2}m-\tfrac{1}{2}\nu) - \ln \tfrac{1}{2}m \right] + O(\kappa^2), \tag{B 8}$$

where (as above) $\nu = 1(0)$ for m odd (even).

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